The Fourier Transform

A little bit of theory, lots of examples and some real life demonstrations

A little bit of theory

Fourier-Transform: $$\mathcal{F}\{f(t)\} = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathrm{i}\omega t}\, dt,$$

Back-Transform: $$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{\mathrm{i}\omega t}\, d\omega.$$

(Wikipedia)

Represents an <u>aperiodic</u> function as the sum of frequency components

A little bit of theory

Fourier-series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cdot \cos(n\omega t) + b_n \cdot \sin(n\omega t))$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\omega t}$$

$$a_n = \frac{2}{T} \int_c^{c+T} f(t) \cdot \cos(n\omega t)\,\mathrm{d}t$$

$$c_n = \frac{1}{T} \int_c^{c+T} f(t) \mathrm{e}^{-\mathrm{i}n\omega t} dt$$

$$b_n = \frac{2}{T} \int_c^{c+T} f(t) \cdot \sin(n\omega t)\,\mathrm{d}t$$

$$\frac{a_0}{2} = \frac{1}{T} \int_c^{c+T} f(t) \mathrm{d}t$$

Represents a <u>periodic</u> function as the sum of frequency components

Examples

Sine and cosine waves with several frequencies and amplitudes (Demo)

Examples

Triangle function, symmetric about origin:

$$f(t) = \frac{4h}{\pi}\left[\cos\omega t + \frac{1}{3^2}\cos 3\omega t + \frac{1}{5^2}\cos 5\omega t + \ldots\right] = \frac{4h}{\pi}\sum_{k=1}^{\infty}\frac{\cos((2k-1)\omega t)}{(2k-1)^2}$$

Triangle function, anti-symmetric about origin:

$$f(t) = \frac{4h}{\pi}\left[\sin\omega t - \frac{1}{3^2}\sin 3\omega t + \frac{1}{5^2}\sin 5\omega t - \ldots\right] = \frac{4h}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin((2k-1)\omega t)}{(2k-1)^2}$$

(Fourier analysis demo)

Examples

Square function, symmetric about origin:

$$f(t) = \frac{4h}{\pi}\left[\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \dots\right] = \frac{4h}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos\left((2k-1)\omega t\right)}{2k-1}$$

Square function, anti-symmetric about origin:

$$f(t) = \frac{4h}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \dots\right] = \frac{4h}{\pi}\sum_{k=1}^{\infty}\frac{\sin\left((2k-1)\omega t\right)}{2k-1}$$

(Fourier analysis demo)

Examples

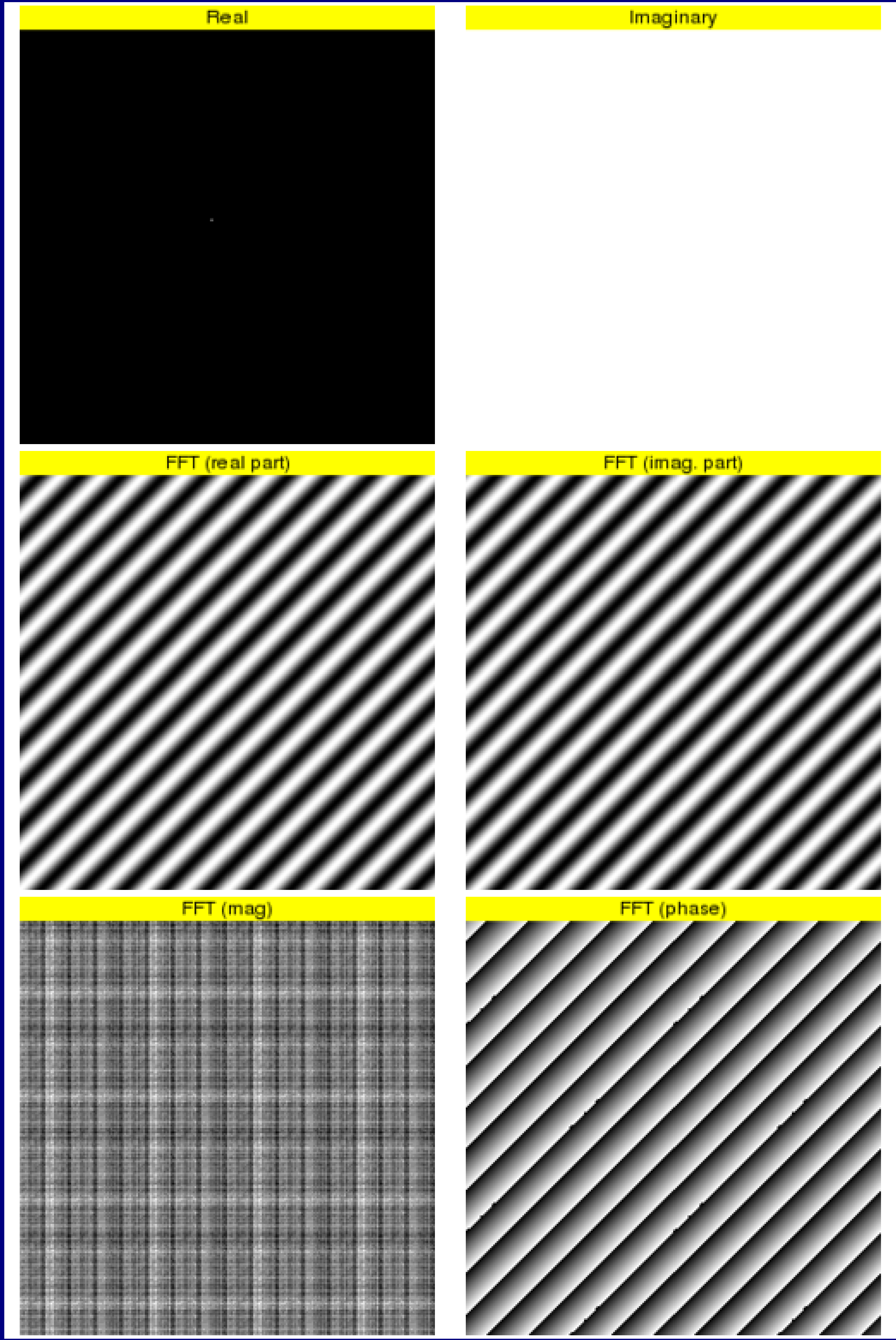
What's this?

$$f(t) = -\frac{2h}{\pi}\left[\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \ldots\right] = -\frac{2h}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin k\omega t}{k}$$

Let's see: (Fourier analysis demo)

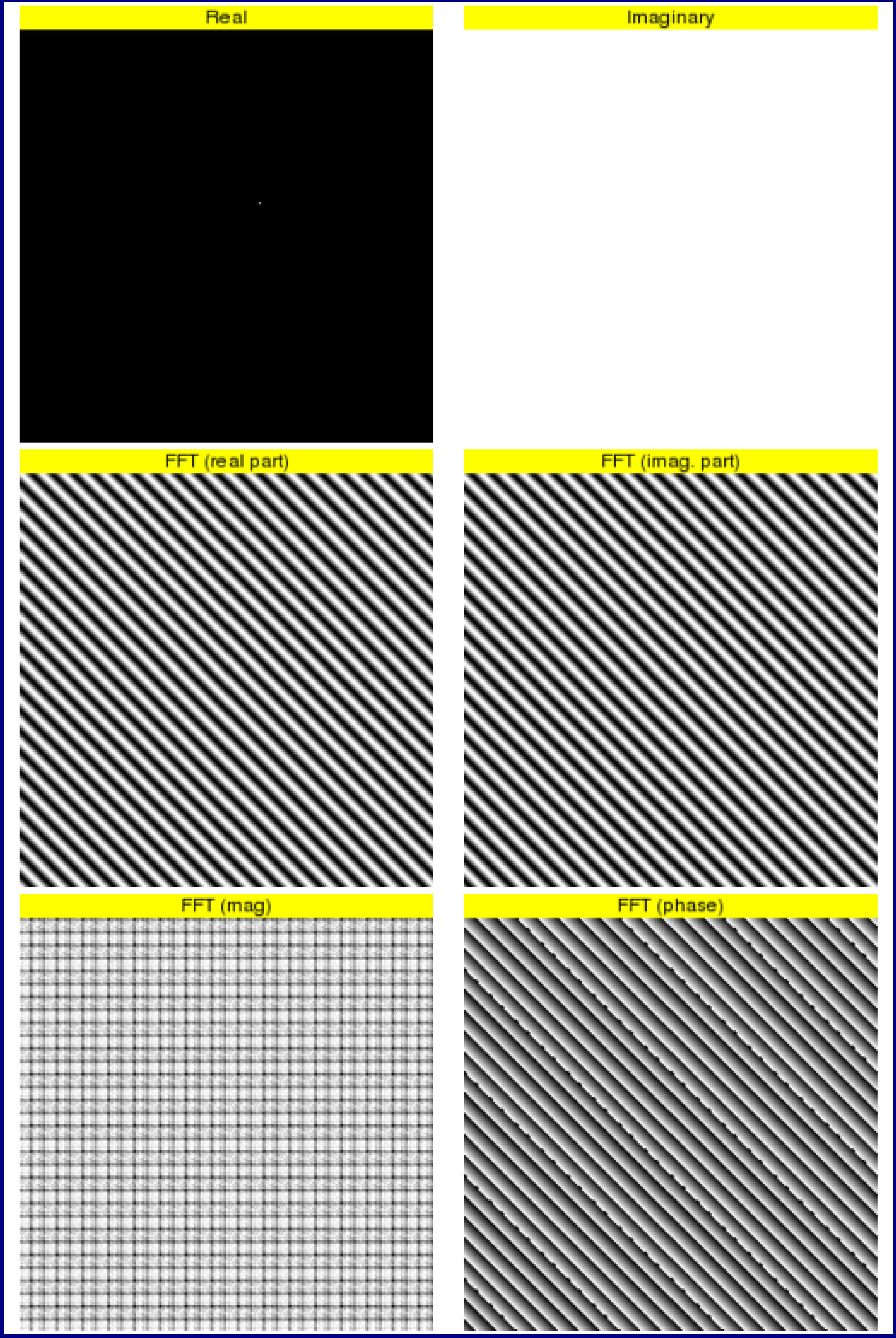
The 2D-case

Offset point source



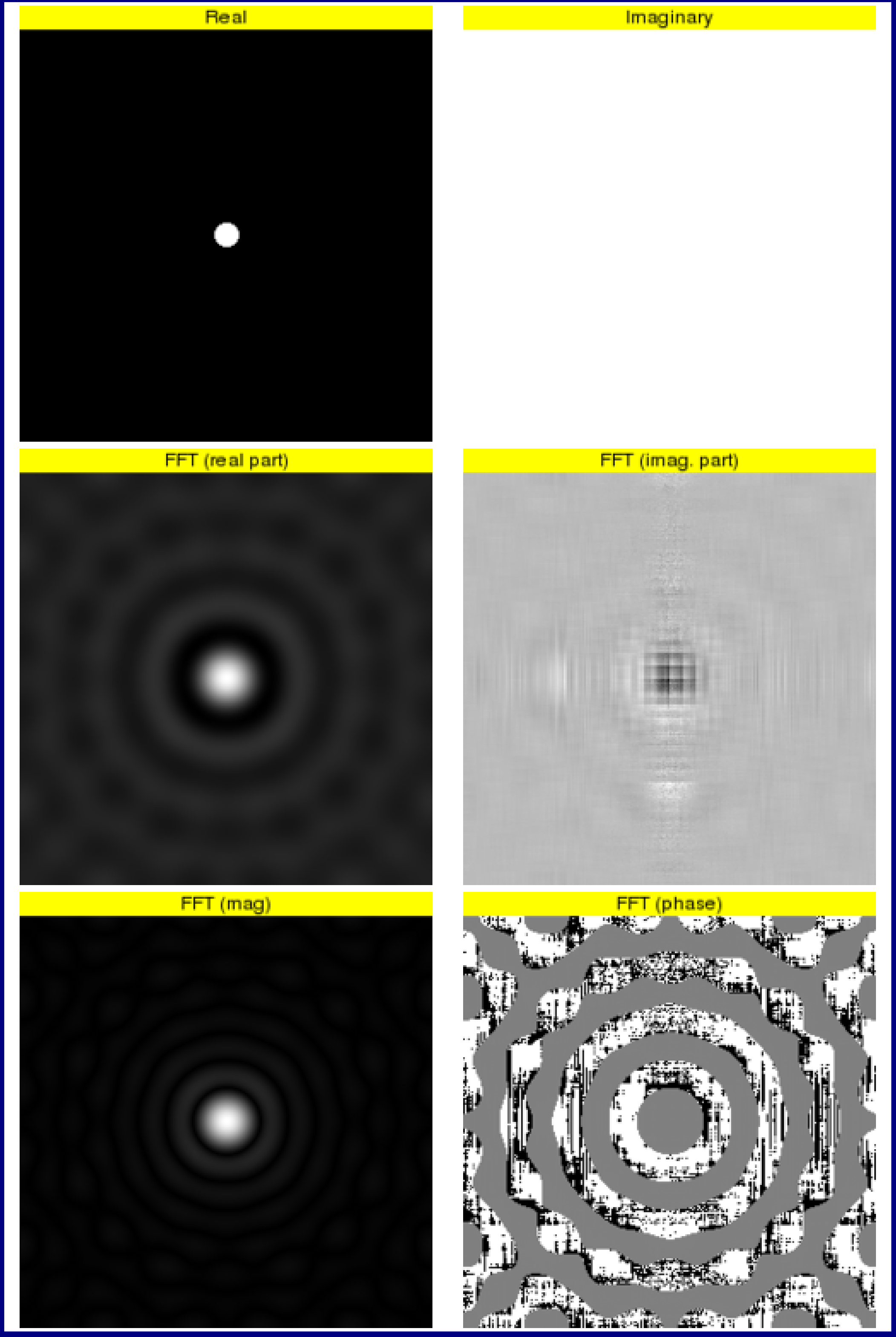
The 2D-case

Offset point source



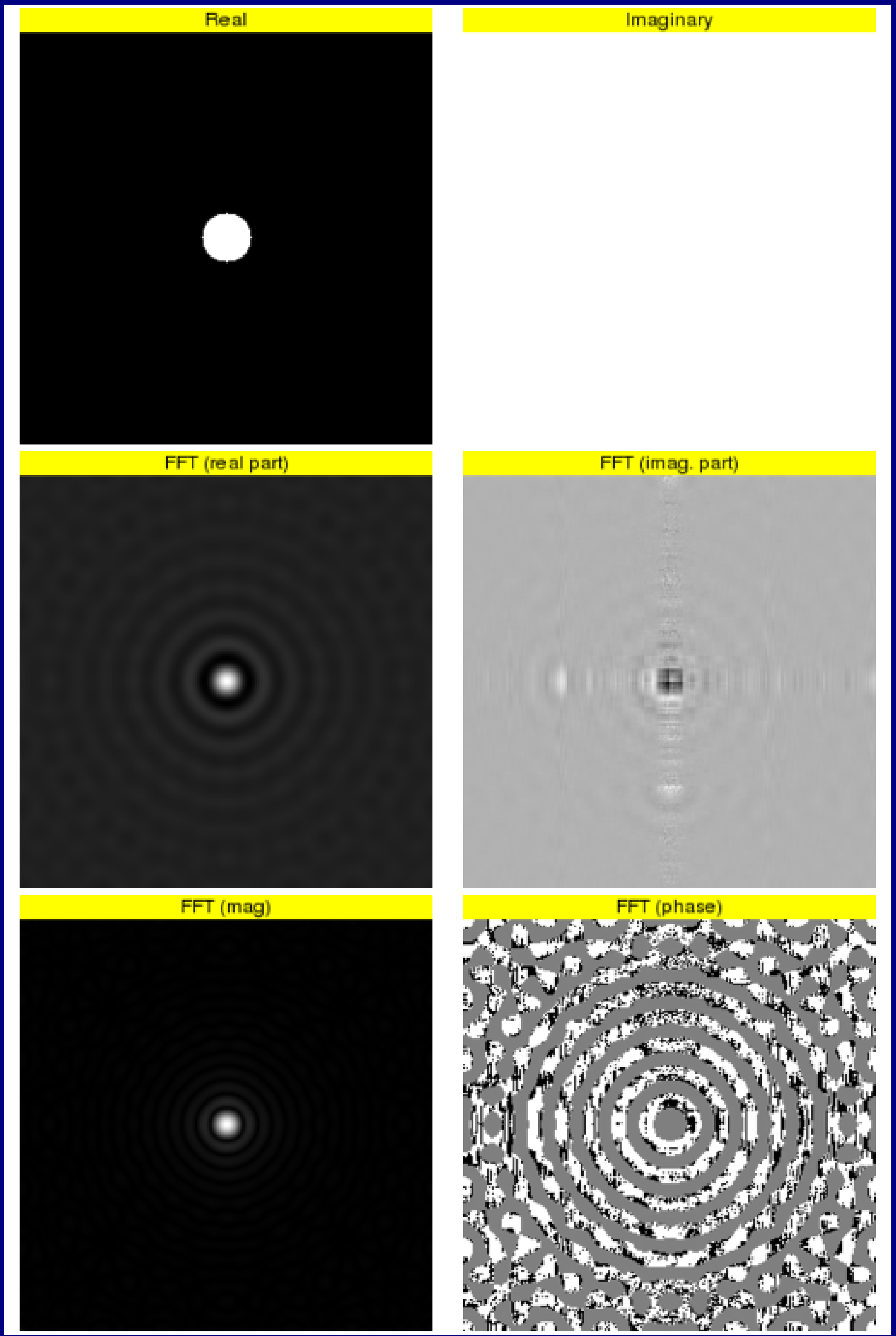
The 2D-case

Uniform disk



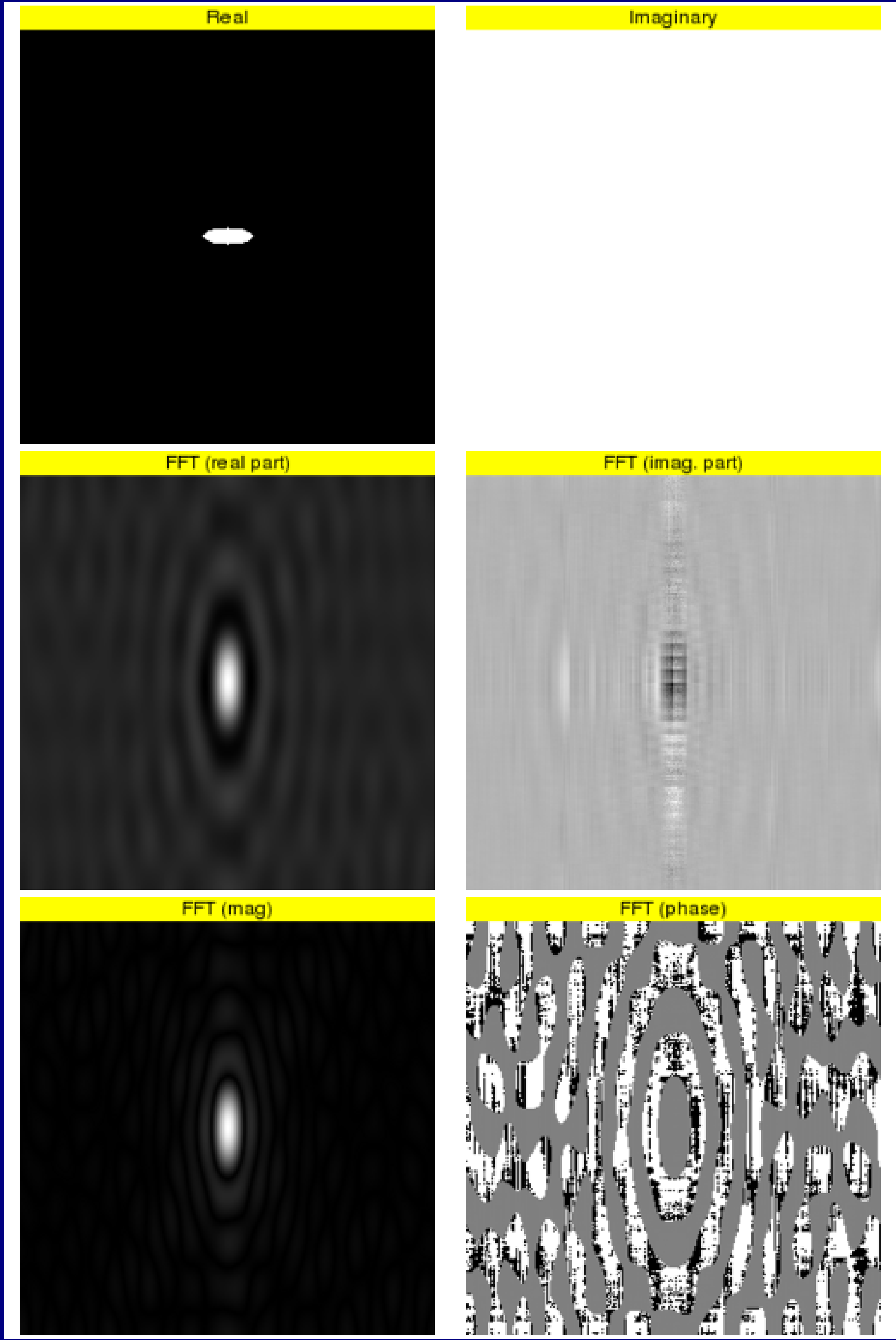
The 2D-case

Uniform disk



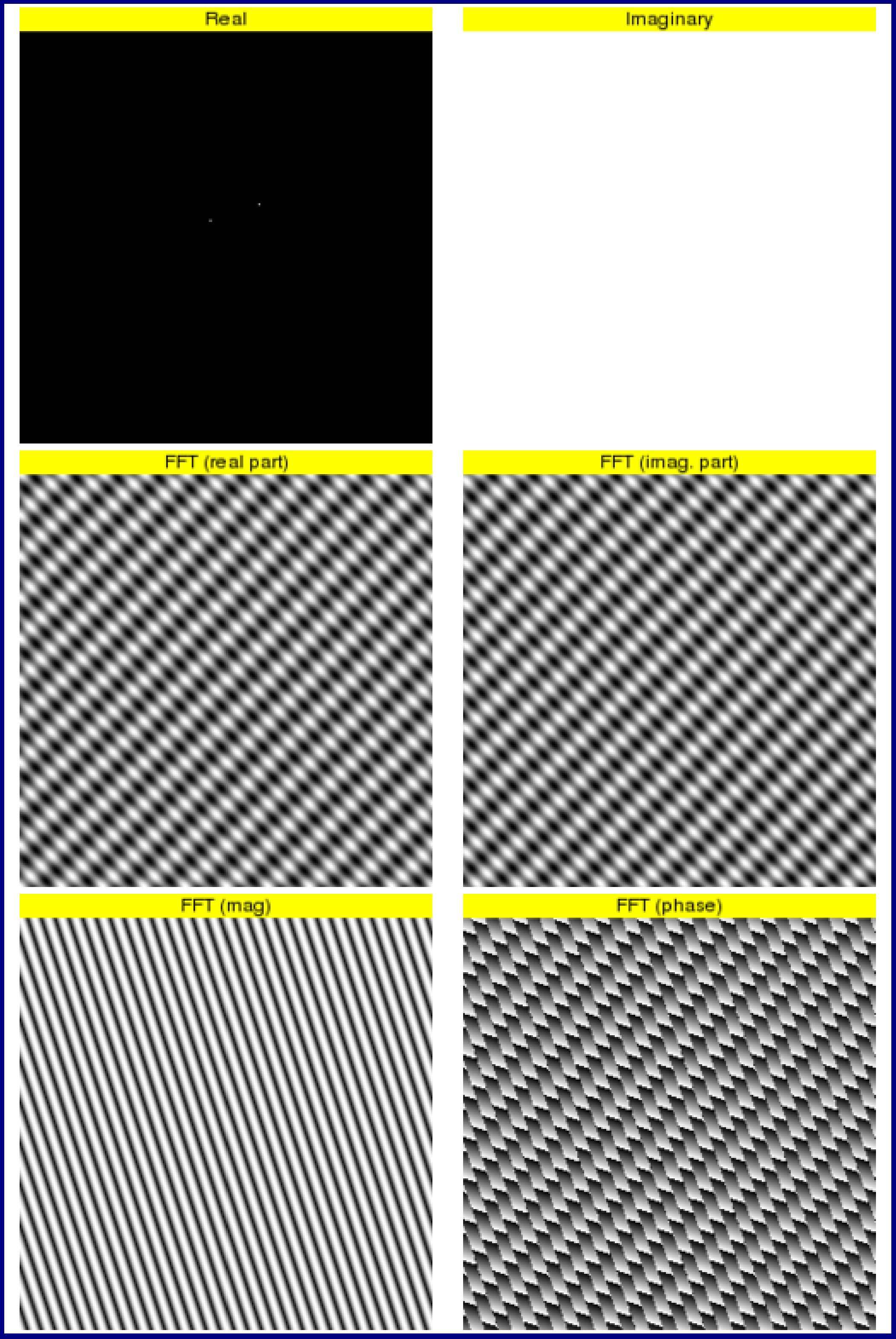
The 2D-case

Elliptical disk



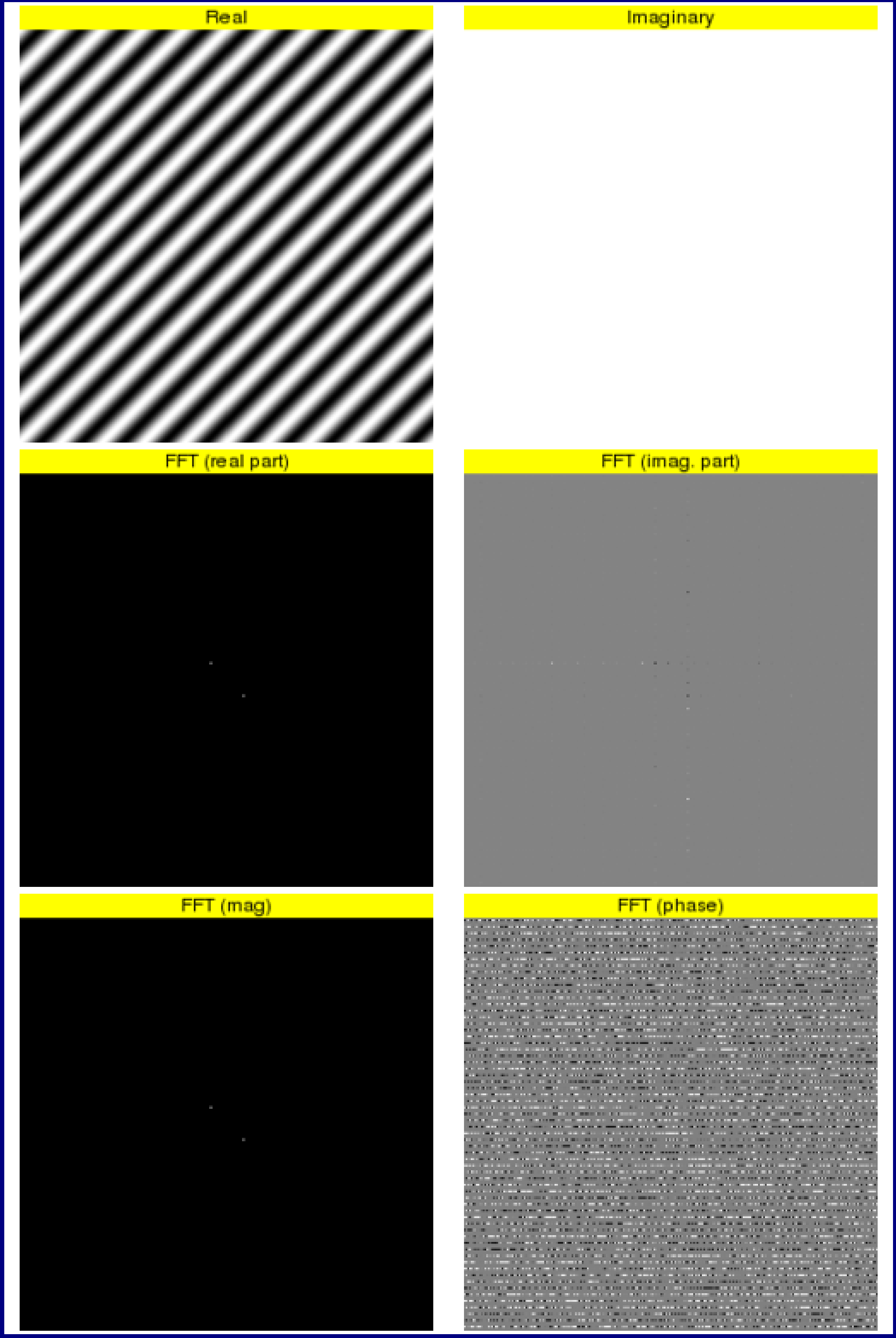
The 2D-case

Two point sources



The 2D-case

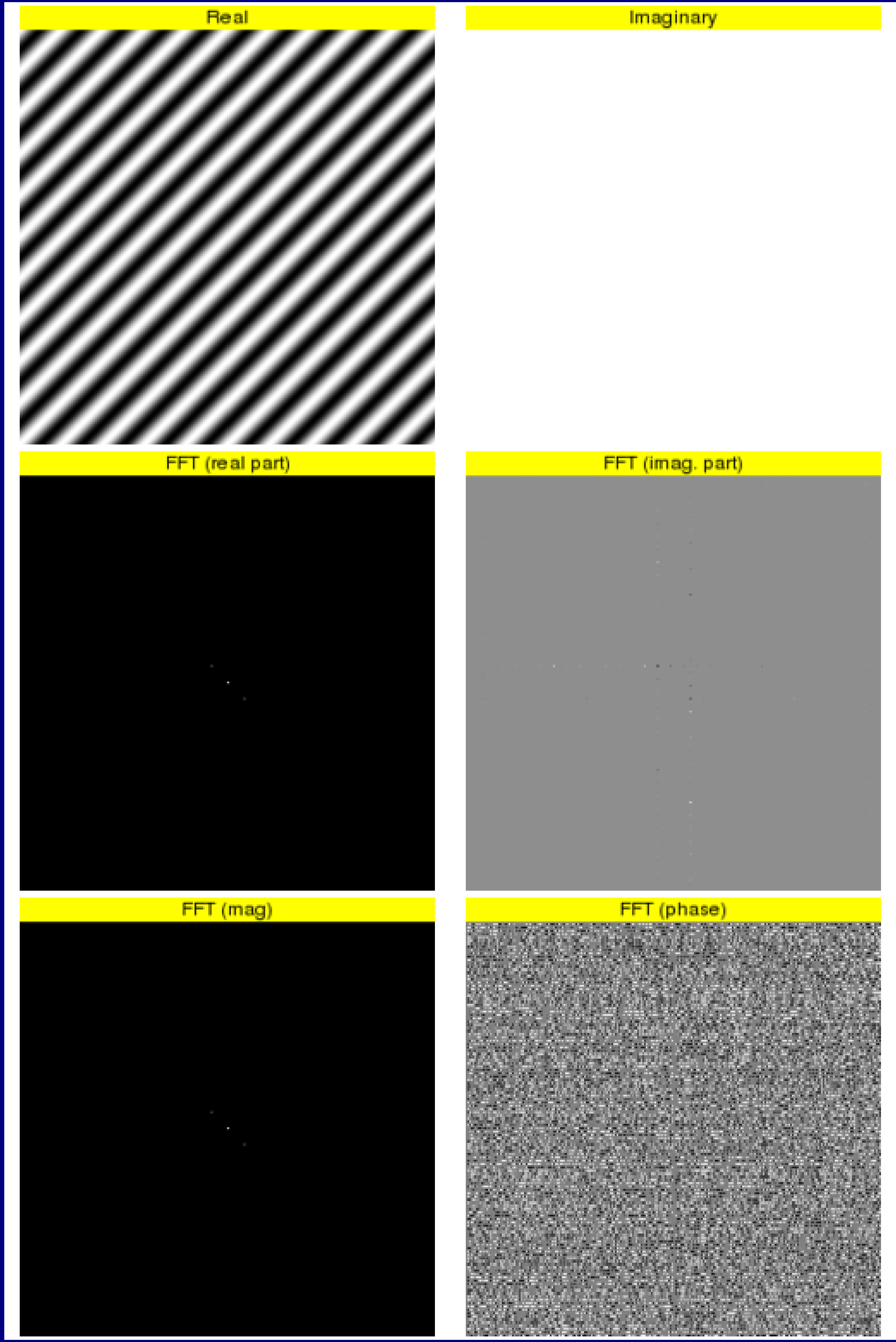
A sine wave



The 2D-case

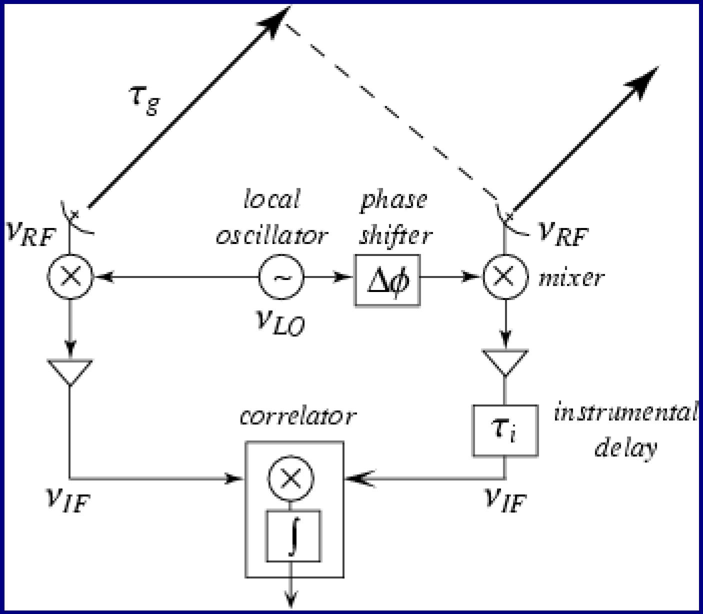
What happened here?

(Photo-demo)



What's the connection to astronomy?

Radio interferometers measure the spatial coherence function:



$$V_\nu(\vec{r}_1, \vec{r}_2) \approx \int I_\nu(\vec{s}) e^{-2\pi i \nu \vec{s}(\vec{r}_1 - \vec{r}_2)/c} d\Omega$$

$$V_\nu(u, v) = \iint I_\nu(l, m) e^{-2\pi i (ul + vm)} dl\, dm$$

(Rippletank-Demo)

What's the connection to astronomy?

Unfortunately, radio interferometers measure only relatively few spatial frequency components (VRI-demo)