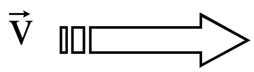
The Virial Theorem

…and maybe a bit on timescales

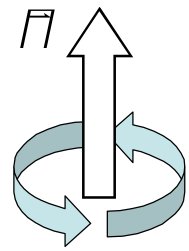
Rudolf Clausius (1822-1888)

- First and second laws of thermodynamics (equivalence of heat & work; heat flows from hot to cold), 1850
- Kinetic theory of gases, 1857 (Über die Art der Bewegung welche wir Wärme nennen)
- Concept of entropy, 1865
- Virial theorem, 1870

Translation

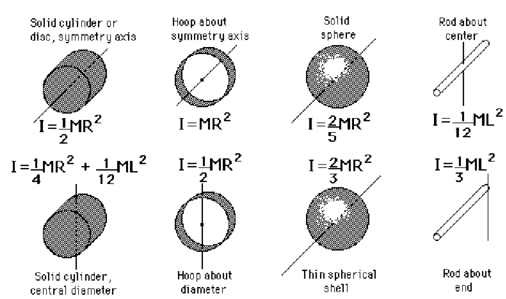


Rotation



Position	$\vec{r}$	$\longleftrightarrow$	θ	Angular position
Velocity	$\vec{v}$	$\longleftrightarrow$	$\vec{\omega}$	Angular velocity
Acceleration	$\vec{a}$	$\longleftrightarrow$	$\vec{\alpha}$	Angular acceleration
Mass	m	$\longleftrightarrow$	I	Moment of inertia

Moment of inertia *(tröghetsmoment)*

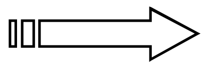


continuous mass distribution: $I = \int_0^M dm\, r^2$

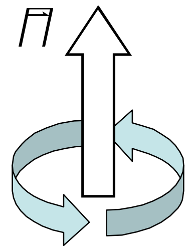
discrete masses: $I = \sum_i m_i r_i^2$

Translation Rotation

$\vec{\mathrm{v}}$



$\vec{\omega}$



Momentum	$\vec{p} = m\vec{\mathrm{v}}$	$\vec{L} = I\vec{\omega}$	Angular momentum
Force	$\vec{F} = m\vec{a}$	$\vec{\tau} = I\vec{\alpha}$	Torque
Kinetic energy	$K = \frac{1}{2}m\mathrm{v}^2$	$K = \frac{1}{2}I\omega^2$	Kinetic energy

Moment of Inertia: Time Derivatives

$$I = \sum_i m_i r^2 = \sum_i m_i \vec{r}_i \cdot \vec{r}_i$$

$$\frac{dI}{dt} = 2\sum_i m_i \dot{\vec{r}}_i \cdot \vec{r}_i$$

$$\boxed{\frac{d^2 I}{dt^2} = 2\sum_i m_i \dot{r}_i^2 + 2\sum_i \vec{F}_i \cdot \vec{r}_i}$$

the "virial"

The Virial of Clausius

$$\sum_i \vec{F}_i \cdot \vec{r}_i = \sum_i \left[\frac{1}{2} \sum_{j \neq i} \vec{F}_{ij} \cdot \vec{r}_i + \frac{1}{2} \sum_{j \neq i} \vec{F}_{ij} \cdot \vec{r}_i \right]$$

$\vec{F}_i = \sum_{j \neq i} \vec{F}_{ij}$

$$\sum_i \vec{F}_i \cdot \vec{r}_i = \frac{1}{2} \sum_i \sum_{j \neq i} \vec{F}_{ij} \cdot \vec{r}_i + \frac{1}{2} \sum_j \sum_{i \neq j} \vec{F}_{ji} \cdot \vec{r}_j$$

$\vec{F}_{ij} = -\vec{F}_{ji}$

$$\boxed{\sum_i \vec{F}_i \cdot \vec{r}_i = \frac{1}{2} \sum_i \sum_{j \neq i} \vec{F}_{ij} \cdot \left(\vec{r}_i - \vec{r}_j \right)}$$

The virial for a self-gravitating system…

$$\sum_i \vec{F}_i \cdot \vec{r}_i = \frac{1}{2} \sum_i \sum_{j \neq i} \vec{F}_{ij} \cdot \left(\vec{r}_i - \vec{r}_j \right)$$

$$\vec{F}_{ij} = -\frac{G m_i m_j}{\left| \vec{r}_i - \vec{r}_j \right|^3} \left(\vec{r}_i - \vec{r}_j \right)$$

$$\sum_i \vec{F}_i \cdot \vec{r}_i = -\frac{1}{2} \sum_i \sum_{j \neq i} \frac{G m_i m_j}{\left| \vec{r}_i - \vec{r}_j \right|} = \Omega$$

…is just its gravitational potential energy

The (Scalar) Virial Theorem

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \sum_i m_i \dot{r}_i^2 + \sum_i \vec{F}_i \cdot \vec{r}_i$$

$$\boxed{\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \Omega}$$

Relation between kinetic and potential energy

Virial Theorem (simplified form)

$$\frac{1}{2}\left\langle \frac{d^2 I}{dt^2} \right\rangle = 2\langle K \rangle + \langle \Omega \rangle$$

time average $\langle Q \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T Q(t)\, dt$

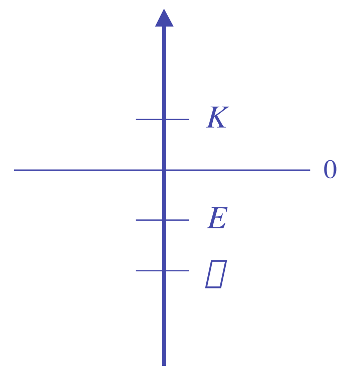
$$\boxed{2\langle K \rangle + \langle \Omega \rangle = 0}$$

$$\langle \Omega \rangle = -2\langle K \rangle$$

Consequences of Virial Theorem

$$\langle E \rangle = \langle K \rangle + \langle \Omega \rangle$$

$$\langle \Omega \rangle = -2 \langle K \rangle$$



What happens when a system (such as a star) contracts?

- Loses energy and heats up (negative heat capacity!)
- Half of energy loss → heats the system
- Half of energy loss → radiated away

Application: Jeans Instability

$$E = K + \Omega < 0$$

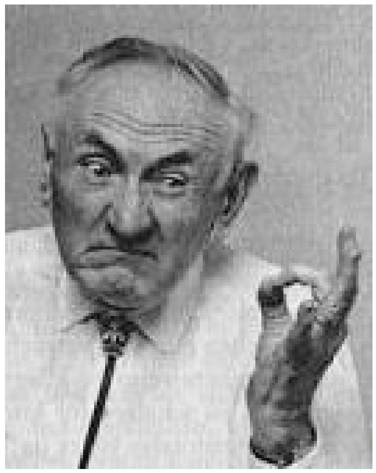
$$K < |\Omega|$$

$$P = nkT$$

$$c_s^2 = P/\rho$$

$$\boxed{R \gtrsim \left(\frac{1}{G\rho_0} \right)^{1/2} c_s}$$

Fritz Zwicky (1898-1974)



- Swiss astronomer who spent many years at Caltech
- Studied the distribution of galaxies in Coma Berenices, 1933
- Insight into nature of extragalactic supernovae; coined the term "neutron star"
- "Supernovae and Cosmic Rays," Zwicky & Baade, 1934
- First to consider gravitational lensing by extragalactic objects, 1937

Masses of clusters

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ON THE MASSES OF NEBULAE AND OF CLUSTERS OF NEBULAE

F. ZWICKY

ABSTRACT

Present estimates of the masses of nebulae are based on observations of the *luminosities* and *internal rotations* of nebulae. It is shown that both these methods are unreliable; that from the observed luminosities of extragalactic systems only lower limits for the values of their masses can be obtained (sec. i), and that from internal rotations alone no determination of the masses of nebulae is possible (sec. ii). The observed internal motions of nebulae can be understood on the basis of a simple mechanical model, some properties of which are discussed. The essential feature is a central core whose internal *viscosity* due to the gravitational interactions of its component masses is so high as to cause it to rotate like a solid body.

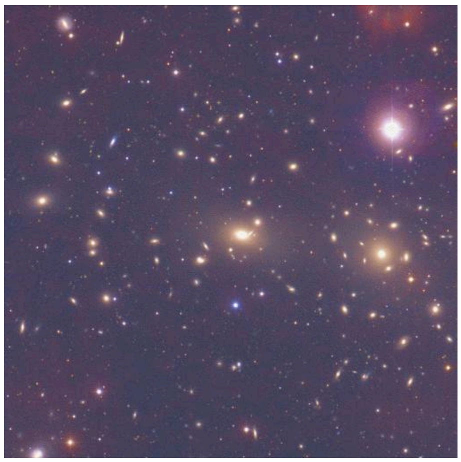
In sections iii, iv, and v three new methods for the determination of nebular masses are discussed, each of which makes use of a different fundamental principle of physics.

Method iii is based on the *virial theorem* of classical mechanics. The application of this theorem to the Coma cluster leads to a minimum value $\overline{M} = 4.5 \times 10^{10} M_{\odot}$ for the average mass of its member nebulae.

Method iv calls for the observation among nebulae of certain *gravitational lens* effects.

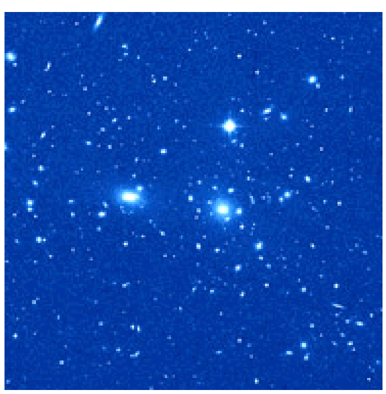
Section v gives a generalization of the principles of ordinary *statistical mechanics* to the whole system of nebulae, which suggests a new and powerful method which ultimately should enable us to determine the masses of all types of nebulae. This method is very flexible and is capable of many modes of application. It is proposed, in particular, to investigate the distribution of nebulae in individual great clusters.

Application: Coma Cluster

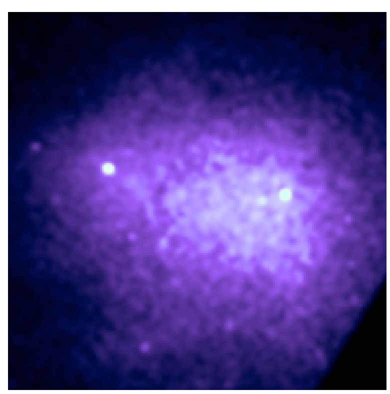


- Cluster of > 1000 galaxies
- Mean redshift: 7000 km/s
- Mean radius (at which projected surface density falls to half of peak value) is 9' on the sky. This is about 0.4 the gravitational radius.
- Dispersion in ***radial*** velocity is about 1020 km/s.
- Like Zwicky, we can use observations + virial theorem to "weigh" the Coma cluster.

IGM and Dark Matter



Optical image



X-ray (Chandra) image

Thoughts on the Virial Theorem

- What did we leave out?
- How about different masses?
- Collisions? Close encounters?
- Timescales